Exact results for average cluster numbers in bond percolation on lattice strips

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We present exact calculations of the average number of connected clusters per site, $\langle k \rangle$, as a function of bond occupation probability p, for the bond percolation problem on infinite-length strips of finite width L_y, of the square, triangular, honeycomb, and kagomé lattices Λ with various boundary conditions. These are used to study the approach of $\langle k \rangle$, for a given p and Λ, to its value on the two-dimensional lattice as the strip width increases. We investigate the singularities of $\langle k \rangle$ in the complex p plane and their influence on the radii of convergence of the Taylor series expansions of $\langle k \rangle$ about $p=0$ and $p=1$.

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I. INTRODUCTION

The study of percolation gives insight into a number of important phenomena such as the passage of fluids through porous media and the effect of lattice defects and disorder on critical phenomena. Here we consider bond percolation. Let $G=G(V,E)$ be a connected graph defined by a set V of vertices (sites) and a set E of edges (bonds) connecting pairs of vertices. We denote the number of vertices and bonds as $n=n(G)=|V|$ and $e(G)=|E|$. In the usual statistical mechanics context, one is interested in a d-dimensional thermodynamic limit of a regular lattice graph Λ in which the bonds are present with probability p. Consider the probability $P(\Lambda,p)$ that a given site belongs to an infinite cluster. For a given lattice Λ, as p decreases from 1, $P(\Lambda,p)$ decreases monotonically until, at a critical value, $p_{c,\Lambda}$, it vanishes and remains identically zero for $0 \le p \le p_{c,\Lambda}$. An interesting quantity in this context is the number of connected components (clusters), including single sites, for a given lattice Λ, divided by the number of sites on the lattice and averaged over all of the graphs in the above ensemble. We denote this mean cluster number per site as $\langle k \rangle_\Lambda$. Reviews on percolation include [1–3].

In this paper we present exact calculations of this average cluster number per site, $\langle k \rangle$, as a function of p, for a variety of infinite-length, finite-width strips of regular lattices [4]. We consider strips of the square, triangular, honeycomb, and kagomé lattices. These are of interest since, at least for modest strip widths, one can obtain explicit analytic expressions for $\langle k \rangle$ and can exactly determine, e.g., singularities that these expressions have in the complex p plane and their influence on series expansions. Our results interpolate between the known exact solutions for the one-dimensional lattice (line) and the case of two dimensions (for which $\langle k \rangle$ is not known exactly as a function of p), and complement numerical simulations and series expansions.

We take the longitudinal and transverse directions to be x and y and denote the size of the lattice strips in these directions as L_x and L_y and the respective boundary conditions as BC_x and BC_y. We focus on the limit of infinite length, $L_x \to \infty$, for which the results are independent of the longitudinal boundary conditions. For an infinite-length strip of a lattice Λ, as the width $L_y \to \infty$, one expects $\langle k \rangle$ to approach a limiting function of p which is independent of the transverse boundary conditions and is equal to $\langle k \rangle$ for the corresponding infinite two-dimensional lattice Λ. In particular, for a given infinite-length, finite-width strip of the lattice Λ, it is of interest to evaluate our exact expressions for $\langle k \rangle$ at $p=p_{c,\Lambda}$ and study how the resultant value approaches the critical value $\langle k \rangle_{c,\Lambda}$ for the corresponding infinite two-dimensional lattice.

For a given graph $G=(V,E)$ we calculate $\langle k \rangle$ by making use of the equations

$$\langle k \rangle_n = \left. \frac{\partial f(G,q,v_p)}{\partial q} \right|_{q=1}, \tag{1.1}$$

where

$$v = v_p \equiv \frac{p}{1-p}, \tag{1.2}$$

$f(G,q,v)$ is the reduced free energy of the q-state Potts model on the graph G [5], and in the limit $n \to \infty$

$$\langle k \rangle = \left. \frac{\partial f(\{G\},q,v_p)}{\partial q} \right|_{q=1}, \tag{1.3}$$

where $\{G\}$ denotes the formal limit $\lim_{n\to\infty} G$ for a given family of graphs. Our method for calculating $\langle k \rangle$ is to use Eq. (1.3) in conjunction with exact results that we have computed for the free energy of the Potts model on infinite-length, finite-width strips of various lattices [6–16].

As background for our exact results, we note some basic properties of $\langle k \rangle$: (i) $\lim_{p\to 0}\langle k \rangle_n = \lim_{p\to 0}\langle k \rangle = 1$; (ii)

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$\lim_{p\to 1}\langle k\rangle_n = 1/n$ and hence $\lim_{p\to 1}\langle k\rangle = 0$; (iii) for a given $\{G\}$, $\langle k\rangle$ is a monotonically decreasing function of p for $p \in [0,1]$. For the infinite-length, finite-width strips considered here, $\langle k\rangle$ is a (real) analytic function of p for $p\in[0,1]$; however, this quantity may have singularities at unphysical values of p, including real values outside the interval $0 \leqslant p \leqslant 1$ and complex values, as will be discussed further below.

The property that $\langle k\rangle$ (calculated in the limit as $L_x \to \infty$) is independent of the longitudinal boundary conditions imposed on the lattice strip follows from the same property for the Potts model free energy. We further expect that, for a given transverse boundary condition and for a given $p \in [0,1]$, the value of $\langle k\rangle$ for an infinite-length strip of the lattice Λ approaches the corresponding value of $\langle k\rangle$ for the (infinite) two-dimensional lattice Λ as $L_y \to \infty$.

Our exact results give insight into a feature of Taylor series expansions in percolation, calculated around $p=0$ and $p=1$, namely, the fact that the radii of convergence of these series expansions around these respective points are typically set by unphysical singularities and are less than the distance from the expansion point to the physical singularity, $p_{c,\Lambda}$. Our results exhibit the same feature: although $\langle k\rangle$ itself is an analytic function of p for $p\in[0,1]$, Taylor series expansions about $p=0$ and $p=1$ typically have radii of convergence less than unity, set by unphysical singularities of $\langle k\rangle$.

II. STRIPS OF THE SQUARE LATTICE

A. $L_y = 1$

The well-known result

$$\langle k\rangle_{1D} = 1 - p \tag{2.1}$$

for the infinite line can be derived directly using probability methods. Here we illustrate how it can be derived via Eq. (1.3). An elementary calculation yields the Potts free energy $f(1D,q,v)=\ln(q+v)$. Using Eq. (1.3) yields the above result for $\langle k\rangle_{1D}$. This has the value 1/2 at $p=p_{c,sq}$ [17] (see Table I).

B. Free transverse boundary conditions

For $L_y \geqslant 2$, we label an infinite-length strip of width L_y of the lattice Λ with given transverse boundary conditions BC_y as $\Lambda,(L_y)_{BC_y}$. In particular, the $L_y=2$ square-lattice strips with free (F) and periodic (P) transverse boundary conditions are denoted $\mathrm{sq},2_F$ and $\mathrm{sq},2_P$.

1. 2_F

The free energy of the Potts model for the $\mathrm{sq},2_F$ strip is [6]

$$f(\mathrm{sq},2_F,q,v) = \frac{1}{2}\ln \lambda_{\mathrm{sq},2F,1} \tag{2.2}$$

where

$$\lambda_{\mathrm{sq},2F,j} = \frac{1}{2}(T_{s2F} \pm \sqrt{R_{s2F}}) \tag{2.3}$$

with $j=1,2$ corresponding to $\pm$ and

TABLE I. Values of $\langle k\rangle$ on infinite-length strips of the lattices Λ (where sq, tri, hc, kag denote square, triangular, honeycomb, and kagomé) of finite width L_y at $p=p_{c,\Lambda}$. The transverse boundary conditions (BC_y) are F and P for free and periodic, respectively. We also include results for the self-dual (sd) strips of the square lattice. The effective coordination number κ_{eff} is defined in Eq. (6.1). The $L_y=\infty$ values of $\langle k\rangle_{p=p_{c,\Lambda}}$ are the values for the two-dimensional lattices Λ [24,25] where these are known exactly, and the dashes for these entries indicate that they do not depend on BC_y.

Λ	BC_y	L_y	κ_{eff}	$\langle k\rangle_{p=p_{c,\Lambda}}$
sq	F	1	2	0.50000
sq	F	2	3	0.28571
sq	F	3	3.33	0.21940
sq	F	4	3.50	0.18753
sq	F	5	3.60	0.16887
sq	P	2	4	0.20000
sq	P	3	4	0.14103
sq	P	4	4	0.12150
sq	P	5	4	0.11284
sq	sd	1	4	0.16667
sq	sd	2	4	0.14407
sq	sd	3	4	0.132545
sq	sd	4	4	0.12561
sq	–	∞	4	0.09808
tri	F	2	4	0.35958
tri	F	3	4.67	0.27149
tri	F	4	5	0.22946
tri	F	5	5.20	0.20491
tri	P	2	6	0.19091
tri	P	3	6	0.14665
tri	P	4	6	0.13138
tri	–	∞	6	0.11184
hc	F	2	2.50	0.20475
hc	F	3	2.67	0.16000
hc	F	4	2.75	0.13834
hc	F	5	2.80	0.12560
hc	P	4	3	0.08983
hc	–	∞	3	0.07687
kag	F	2	3.2	0.22918
kag	F	3	3.5	0.17220
kag	P	2	4	0.11149

$$T_{s2F} = v^3 + 4v^2 + 3qv + q^2, \tag{2.4}$$

$$R_{s2F} = v^6 + 4v^5 - 2qv^4 - 2q^2v^3 + 12v^4 + 16qv^3 + 13q^2v^2 + 6q^3v + q^4. \tag{2.5}$$

In Eq. (2.3) only the $j=1$ term is relevant for the free energy, while the $j=2$ term will be discussed below.

From Eq. (2.2) we calculate the average cluster number per site

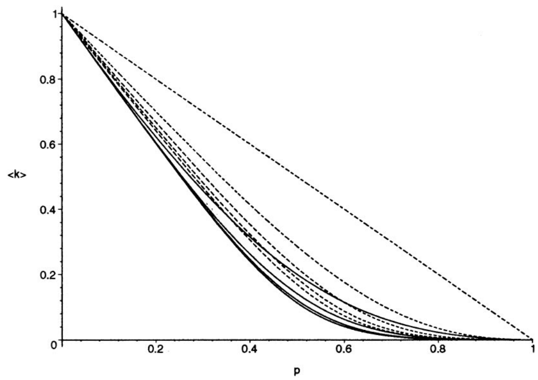

FIG. 1. Plots of $\langle k\rangle$ (vertical axis) as a function of $p\in[0,1]$ (horizontal axis) for infinite-length, finite-width strips of the square lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k\rangle$, for $1_F \le (L_y)_F \le 5_F$, and the solid curves are, in the same order, for $2_P \le (L_y)_P \le 5_P$.

$$\langle k\rangle_{\mathrm{sq},2_F} = \frac{(1-p)^2(2+p-2p^2)}{2(1-p^2+p^3)}. \tag{2.6}$$

This is plotted in Fig. 1 together with cluster numbers calculated for other strips. At $p=p_{c,\mathrm{sq}}=1/2$, this average cluster number has the value $\langle k\rangle_{\mathrm{sq},2_F}=2/7\simeq 0.285\,71$.

From the exact expression for $\langle k\rangle_{\mathrm{sq},2_F}$ we compute the respective Taylor series expansions

$$\langle k\rangle_{\mathrm{sq},2_F} = 1 - \frac{3}{2}p + \frac{1}{2}p^4 + \frac{1}{2}p^6 + O(p^7) \quad \text{for} \quad p \to 0 \tag{2.7}$$

and, in terms of the variable $r=1-p$,

$$\langle k\rangle_{\mathrm{sq},2_F} = \frac{1}{2}r^2 + 2r^3 - \frac{7}{2}r^5 + O(r^6) \quad \text{for} \quad r \to 0. \tag{2.8}$$

Thus for this strip $\langle k\rangle$ is linear for small p and vanishes quadratically as $p\to 1$. As expected for such a small width, these series differ from the series for the (infinite) square lattice, although the linear behavior for small p is common to both.

The expression for $\langle k\rangle$ for this strip has singularities, which are simple poles, where the denominator $1-p^2+p^3=0$, at

$$p \simeq -0.7549, \quad 0.8774 \pm 0.7449i. \tag{2.9}$$

The first of these poles is the closest to the origin and determines the radius of convergence of the small-p Taylor series in Eq. (2.7) to be approximately 0.7549. The complex pair are the same distance from the point $p=1$ and imply that this series converges for $|1-p| \lesssim 0.7549$. Thus, although $\langle k\rangle_{\mathrm{sq},2_F}$ is an analytic function of p for $p\in[0,1]$, the Taylor series expansions about $p=0$ and $p=1$ have radii of convergence less than unity because of singularities of this function at real and complex values outside the physical interval $0\le p\le 1$. It is interesting that although $\lambda_{\mathrm{sq},2_F}$ is an algebraic function of v [and hence p, via Eq. (1.2)], the resultant expression for $\langle k\rangle_{\mathrm{sq},2_F}$ is a rational function of p. However, this is a consequence of the small value of L_y. The same comment applies to the property that $\langle k\rangle_{\mathrm{sq},2_F}$ is meromorphic, i.e., its only singularities are simple poles. As will be seen, these features are also true of the cluster number $\langle k\rangle$ for other $L_y=2$ strips considered here.

In this very simple context of a quasi-one-dimensional (quasi-1D) strip, one hence gains some insight into the similar influence of unphysical singularities in series expansions about $p=0$ and $p=1$ for percolation on higher-dimensional lattices. To understand these poles more deeply, we observe that although the free energy $f(\mathrm{sq},2_F,q,v)$ depends only on $\lambda_{\mathrm{sq},2F,1}$, the partition function for free longitudinal boundary conditions [18] and $q\neq 1$ in general is a symmetric sum of L_xth powers of both of the $\lambda_{\mathrm{sq},2F,j}$'s for $j=1$ and $j=2$ [given as Eq. (5.17) of Ref. [6]]. We are interested in the limit $q\to 1$. It is necessary to take account of a subtlety concerning the dependence of the complex- v phase boundary $\mathcal{B}$ of the Potts model, as a function of q. In previous work [see Eqs. (2.8)–(2.12) of Ref. [6] and Eq. (1.9) of Ref. [19]] we pointed out the noncommutativity at certain special values of q_s, including $q_s=0,1$, namely

$$\lim_{n\to\infty}\lim_{q\to q_s} Z(G,q,v)^{1/n} \neq \lim_{q\to q_s}\lim_{n\to\infty} Z(G,q,v)^{1/n} \tag{2.10}$$

and we noted that, because of this noncommutativity, for the special set of points $q=q_s$ one must distinguish between (i) $(\mathcal{B}(\{G\},q_s))_{nq}$, the continuous accumulation set of the zeros of $Z(G,q,v)$ obtained by first setting $q=q_s$ and then taking $n\to\infty$, and (ii) $(\mathcal{B}(\{G\},q_s))_{qn}$, the continuous accumulation set of the zeros of $Z(G,q,v)$ obtained by first taking $n\to\infty$, and then taking $q\to q_s$. For these special points,

$$(\mathcal{B}(\{G\},q_s))_{nq} \neq (\mathcal{B}(\{G\},q_s))_{qn}. \tag{2.11}$$

A previous case of this was the $q=2$ (Ising) special case of the Potts model. Indeed, in that case it was noted that $\mathcal{B}_{qn}$ does not have the inversion symmetry $e^K\to e^{-K}$ that characterizes the Ising model and its complex-temperature phase boundary $\mathcal{B}_{nq}$ for a bipartite lattice (see pp. 396, 433–435 of Ref. [6]). This noncommutativity is also present at the value $q=1$ relevant for percolation. If one uses the definition $\mathcal{B}_{nq}$ with $q=1$ for percolation, as one uses the definition $\mathcal{B}_{nq}$ with $q=2$ for the Ising model, then while $\mathcal{B}_{nq}$ is nontrivial for the Ising model, $\mathcal{B}_{nq}$ is trivial for the percolation problem. The reason for this is that if one sets $q=1$ first, then, from the Hamiltonian definition of the Potts model, since the spins are the same on all sites, the spin-spin interactions on each bond contribute a factor e^K to the partition function, so one has the elementary result

$$Z(G,1,v) = e^{K\,e(G)} = (v+1)^{e(G)}. \tag{2.12}$$

Substituting $v=v_p$ as in Eq. (1.2) gives

$$Z(G,1,v_p) = (1-p)^{-e(G)}. \tag{2.13}$$

Evidently, $Z(G,1,v_p)$ has no zeros, so that $\mathcal{B}_{nq}=\emptyset$ in the complex p plane. Equivalently, $Z(G,1,v)$ has only a single zero at the point $v=-1$, which maps, via Eq. (1.2), to the circle at infinity in the complex p plane. We have noted

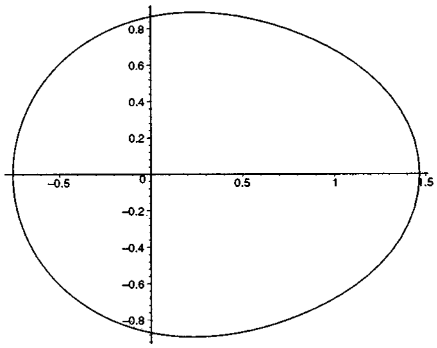

FIG. 2. Plot of the boundary $\mathcal{B}_{qn}$ in the complex- p plane for the infinite-length sq, 2_F lattice strip. Horizontal and vertical axes are Re(p) and Im(p).

above that, in general, for $q \neq 1$, the partition function of the Potts model consists of a symmetric sum of L_xth powers of $\lambda_{sq,2_F,1}$ and $\lambda_{sq,2_F,2}$; if one sets $q=1$, the coefficient of $(\lambda_{sq,2_F,2})^{L_x}$ vanishes, and the $(\lambda_{sq,2_F,1})^{L_x}$ term, with its coefficient, reduces to the form (2.13) (where in the labeling convention of Ref. [6], L_x+1 denotes the number of squares on the sq, 2_F strip).

However, the fact that Eq. (1.3) involves a derivative means that it is sensitive to properties of the Potts model in the neighborhood of the point $p=1$ as well as at this point. This suggests that one consider the possible role of the locus $\mathcal{B}_{qn}$, although one must use caution in doing this because of the noncommutativity discussed above. Below we shall use the notation $\mathcal{B}_{qn}$ to mean specifically the boundary defined for $n \to \infty$ and $q \to 1$, relevant to the percolation problem.

We find some intriguing connections between the locus $\mathcal{B}_{qn}$ and complex- p singularities in $\langle k \rangle$. Let us first calculate $\mathcal{B}_{qn}$ for the sq, 2_F strip in the limit $q \to 1$. Evaluating Eq. (2.3) for $q \to 1$, we obtain

$$\lambda_{sq,2F,1} = \frac{1}{(1-p)^3} \tag{2.14}$$

and

$$\lambda_{sq,2F,2} = \frac{p^2}{(1-p)^2}. \tag{2.15}$$

The locus $\mathcal{B}_{qn}$ is the set of solutions of the equation of degeneracy in magnitude of dominant λ's. This locus can be seen as a special case of the more general phase boundary for the Potts model in the v plane for a fixed q, or in the q plane for a fixed v. For the present case, since there are only two $\lambda_{sq,2F,j}$'s, for $j=1,2$, this equation is $|\lambda_{sq,2F,1}|=|\lambda_{sq,2F,2}|$, i.e.,

$$|p^2(1-p)| = 1. \tag{2.16}$$

In terms of the polar coordinates $p=\rho e^{i\theta}$ this equation reads $\rho^4(1+\rho^2-2\rho\cos\theta)=1$. The solution is a closed egg-shaped curve, shown in Fig. 2, that crosses the real-p axis at $p \simeq -0.7549$ and $p \simeq 1.466$ and the imaginary-p axis at $p \simeq \pm 0.8688i$. This thus constitutes the phase boundary in the complex-p plane, separating this plane into two regions. As follows from the general discussion above, the physical interval $0 \leqslant p \leqslant 1$ lies entirely in one phase. The three poles of $\langle k \rangle_{sq,2_F}$ listed in Eq. (2.9) lie on this boundary $\mathcal{B}_{qn}$.

2. $3_F, 4_F, 5_F$

The Potts model free energy f for the infinite-length sq, 3_F strip was calculated in Ref. [7]. The free energy is given by $f(\mathrm{sq},3_F,q,v)=(1/3)\ln\lambda_{sq,3_F}$, where $\lambda_{sq,3_F}$ is the (maximal) root of an algebraic equation of degree 4. Because of the complicated nature of the expression for this quartic root, we do not present it here. We have calculated $f(\mathrm{sq},3_F,q,v)$, and hence $\langle k \rangle_{sq,3_F}$, to high precision by numerically solving for $f(\mathrm{sq},3_F,q,v)$ for a range of values of q near unity, for each value of p, and carrying out the differentiation in Eq. (1.3). Although this is numerical, the computational steps can be carried out with almost arbitrarily high precision, so that, in practice, it is essentially equivalent to evaluating an explicit exact analytic expression. We also apply this procedure for larger strip widths, using the exact calculation of f for sq, 4_F and sq, 5_F in Ref. [14] (see also [16]). The resulting values of $\langle k \rangle$ are plotted as functions of p in Fig. 1, and the values of $\langle k \rangle$ at $p=p_{c,sq}=1/2$ are listed in Table I. One could carry out similar calculations of $\langle k \rangle$ for larger values of L_y, but our results are sufficient to show the nature of the approach of $\langle k \rangle$ on these infinite-length, finite-width strips to the average cluster number for the corresponding infinite two-dimensional lattice. Indeed, one of the most interesting pieces of information that we get from our results, the exact determination of singularities of $\langle k \rangle$ in the complex-p plane and their effect on the radii of convergence of series expansions, can be obtained only for strip widths that are small enough so that we can get exact explicit analytical forms for $\langle k \rangle$.

C. Periodic transverse boundary conditions

1. 2_P

By using periodic transverse boundary conditions, one minimizes finite-width effects in this transverse direction. We consider first the sq, 2_P strip. Note that this strip has double transverse bonds. The free energy was computed in Ref. [7] and is given by

$$f(\mathrm{sq},2_P,q,v) = \frac{1}{2}\ln\lambda_{sq,2P,1} \tag{2.17}$$

where

$$\lambda_{sq,2P,j} = \frac{1}{2}(T_{s2P} \pm \sqrt{R_{s2P}}) \tag{2.18}$$

with $j=1,2$ corresponding to $\pm$ and

$$T_{s2P} = 6v^2 + 4qv + q^2 + 4v^3 + qv^2 + v^4 \tag{2.19}$$

and

$$R_{s2P} = (v^4 + 6v^3 + 8v^2 + 3qv^2 + 6qv + q^2) \times (v^4 + 2v^3 + 4v^2 - qv^2 + 2qv + q^2). \tag{2.20}$$

From this, using Eq. (1.3), we calculate

$$\langle k\rangle_{\mathrm{sq},2_P} = \frac{(1-p)^2(2-3p^2+2p^3)}{2(1+p-p^2)(1-p+p^2)}. \quad (2.21)$$

At the value of $p_{c,\mathrm{sq}}=1/2$ for the infinite square lattice this has the value $\langle k\rangle_{\mathrm{sq},2_P}=1/5$.

The expression (2.21) has poles where $1+p-p^2$ vanishes, at

$$p_{1,2} = \frac{1}{2}(1 \pm \sqrt{5}) \simeq 1.618, \quad -0.6180, \quad (2.22)$$

and where $1-p+p^2$ vanishes, at

$$p_{3,4} = \frac{1}{2}(1 \pm \sqrt{3}i) \simeq 0.5 \pm 0.866i. \quad (2.23)$$

The second and first of these poles are closest to the points $p=0$ and $p=1$ and determine the radii of convergence of the respective Taylor series expansions about these points both to be 0.618. These expansions are

$$\langle k\rangle_{\mathrm{sq},2_P} = 1 - 2p + \frac{1}{2}p^2 + 2p^4 + O(p^5) \quad (2.24)$$

and

$$\langle k\rangle_{\mathrm{sq},2_P} = \frac{1}{2}r^2 + 2r^4 - 2r^5 + O(r^6). \quad (2.25)$$

Note that, in accord with the fact that the coordination number of this and any infinite-length lattice strip of the square lattice with periodic transverse boundary conditions is 4, the coefficient of the linear term in the small-p expansion is equal to that of the expansion for the full square lattice.

We next discuss the connection of the poles in $\langle k\rangle_{\mathrm{sq},2_P}$ with the locus $\mathcal{B}_{qn}$. Although $f(\mathrm{sq},2_P,q,v)$ depends only on the quantity $\lambda_{\mathrm{sq},2P,1}$, the Potts model partition function for $q\neq 1$ involves a symmetric sum of L_xth powers of both $\lambda_{\mathrm{sq},2P,1}$ and $\lambda_{\mathrm{sq},2P,2}$ [14]. Evaluating $\lambda_{\mathrm{sq},2P,j}$ for $q\to 1$, we have

$$\lambda_{\mathrm{sq},2P,1} = \frac{1}{(1-p)^4}, \quad (2.26)$$

$$\lambda_{\mathrm{sq},2P,2} = \frac{p^2}{(1-p)^2}. \quad (2.27)$$

The locus $\mathcal{B}_{qn}$ is the set of solutions of the equation $|\lambda_{\mathrm{sq},2P,1}|=|\lambda_{\mathrm{sq},2P,2}|$, i.e.,

$$|p(1-p)| = 1. \quad (2.28)$$

In terms of the polar coordinates defined above, this equation reads $\rho^2(1+\rho^2-2\rho\cos\theta)=1$. The solution forms a closed oval curve in the complex- p plane, shown in Fig. 3, that crosses the real axis at the points $p_{1,2}$ in Eq. (2.22) and the imaginary axis at the points $p\simeq \pm 0.786\,15i$. As in the case of the $\mathrm{sq},2_F$ strip, this curve separates the p plane into two regions. All of the four poles of $\langle k\rangle_{\mathrm{sq},2_P}$ given in Eqs. (2.22) and (2.23) lie on this curve $\mathcal{B}_{qn}$. This property—that the singularities of $\langle k\rangle$ lie on $\mathcal{B}_{qn}$—is analogous to the property that the singularities of thermodynamic functions of spin models lie on the complex-temperature phase boundaries for these models, as we have studied in earlier work [20–22]. Having pointed out the connection between these singularities and the locus $\mathcal{B}_{qn}$, we shall, for the strips considered below, just summarize the singularities of $\langle k\rangle$.

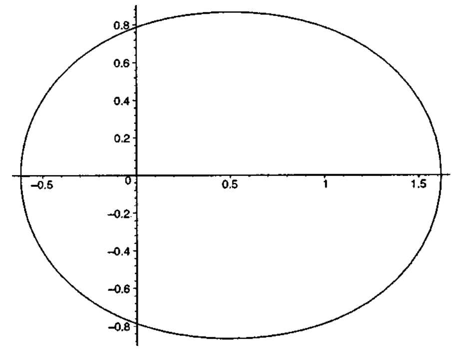

FIG. 3. Plot of the boundary $\mathcal{B}_{qn}$ in the complex- p plane for the infinite-length $\mathrm{sq},2_P$ lattice strip. Horizontal and vertical axes are $\mathrm{Re}(p)$ and $\mathrm{Im}(p)$.

2. $3_P, 4_P, 5_P$

For the $\mathrm{sq},3_P$ strip, $f(\mathrm{sq},3_P,q,v)=(1/3)\ln\lambda_{\mathrm{sq},3_P}$, where $\lambda_{\mathrm{sq},3_P}$ is the (maximal) root of a cubic equation. Although it is possible to display an analytic result for $\langle k\rangle_{\mathrm{sq},3_P}$, it is sufficiently cumbersome that we do not give it here. It is an algebraic, rather than rational, function of p. We do display the small- p expansion, which is

$$\langle k\rangle_{\mathrm{sq},3P} = 1 - 2p + \frac{1}{3}p^3 + p^4 + O(p^5). \quad (2.29)$$

The free energy f was calculated for the $\mathrm{sq},4_P$ and $\mathrm{sq},5_P$ strips in Ref. [14], and $\lambda_{\mathrm{sq},4_P}$ and λ_{5_P} are roots of equations of too high a degree to allow an explicit analytic solution. Accordingly, we compute $\langle k\rangle$ by the numerical procedure discussed above. Results are given in Fig. 1 and Table I.

D. Self-dual strips of the square lattice

It is of interest to calculate $\langle k\rangle$ for strips of the square lattice that maintain a property of the infinite square lattice, namely, self-duality. The strips with free and periodic transverse boundary conditions considered above are not self-dual. However, one can construct a cyclic strip that is self-dual by adding a single external site to a cyclic square-lattice strip of width L_y and then adding bonds connecting all of the sites on one side of the strip to this single external site. We denote a self-dual (sd) strip of this type as $\mathrm{sq},(L_y)_{\mathrm{sd}}$. Before presenting our calculations, a remark is in order concerning p_c for these strips. The physical meaning of p_c for a usual infinite lattice is, as mentioned before, that for $p\geq p_c$ there exists a percolation cluster linking two points that are an arbitrarily large distance apart. Now consider the simplest of the cyclic self-dual lattice graphs, with $L_y=1$; this is a wheel graph, having a rim forming a circuit and a central site ($\sim$ axle) connected to the sites on the rim by L_x bonds forming spokes. Evidently, even in the limit $L_x\to\infty$, the maximum distance between any two sites on this lattice graph is two bonds; to get from any site on the rim to any other site, one takes a minimum-distance route that goes inward along one spoke to the central site and out again on another spoke to the other site. Similarly, for any finite L_y, even as $L_x \to \infty$ there is a maximal finite distance $2L_y$ bonds between any two sites. Therefore, although this family of cyclic lattice strips does maintain the property of self-duality of the infinite square lattice, the notion of a critical p_c beyond which there is a percolation cluster linking two sites arbitrarily apart is not applicable to it since no sites are arbitrarily far apart.

1. 1_{sd}

The free energy is [10,13]

$$f(\mathrm{sq}, 1_{\mathrm{sd}}, q, v) = \ln \lambda_{\mathrm{sd}1} \tag{2.30}$$

where

$$\lambda_{\mathrm{sd}1} = \frac{1}{2}(T_{\mathrm{sd}1} + \sqrt{R_{\mathrm{sd}1}}) \tag{2.31}$$

with

$$T_{\mathrm{sd}1} = 3v + q + v^2, \tag{2.32}$$

$$R_{\mathrm{sd}1} = 5v^2 + 2vq + 2v^3 + q^2 - 2v^2 q + v^4. \tag{2.33}$$

From this we calculate

$$\langle k \rangle_{\mathrm{sq},1_{\mathrm{sd}}} = \frac{(1-p)^3}{1-p+p^2}. \tag{2.34}$$

We have $\langle k \rangle_{\mathrm{sq},1_{\mathrm{sd}}} = 1/6$ at $p = p_{c,\mathrm{sq}}$. The mean cluster number $\langle k \rangle$ in Eq. (2.34) has the following Taylor series expansions for $p \to 0$ and $p \to 1$:

$$\langle k \rangle_{\mathrm{sq},1_{\mathrm{sd}}} = 1 - 2p + p^3 + p^4 - p^6 - p^7 + p^9 + O(p^{10}), \tag{2.35}$$

$$\langle k \rangle_{\mathrm{sq},1_{\mathrm{sd}}} = r^3 + r^4 - r^6 - r^7 + r^9 + O(r^{10}). \tag{2.36}$$

One sees that the coefficient of the linear term in the small- p expansion correctly matches that of the series for the infinite square lattice and the power of the leading-order term in $p \to 1$ expansion is 3, which, although not equal to the power 4 in the corresponding expansion about $p=1$, is at least closer than the power of 2 for the $L_y=2$ square-lattice strips with free or periodic boundary conditions. The poles in Eq. (2.34) at $p = (1/2)(1 \pm \sqrt{3}i)$ set the radii of convergence of the small- p and small- r expansions as unity in both cases, i.e., the full physical interval $0 \le p \le 1$.

2. $2_{sd}, 3_{sd}, 4_{sd}$

For these strips, the free energy has the form [10,13] $f(\mathrm{sq}, (L_y)_{\mathrm{sd}}, q, v) = (1/L_y) \ln \lambda_{\mathrm{sq},(L_y)_{\mathrm{sd}}}$, where $\lambda_{\mathrm{sq},(L_y)_{\mathrm{sd}}}$ are maximal roots of algebraic equations of degree 5 or higher. Hence, it is thus not possible to obtain a closed-form analytic solution for this root. We thus follow the same high-precision numerical procedure as described above (see Fig. 4).

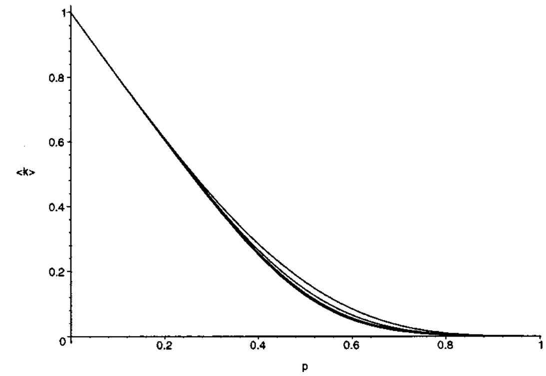

FIG. 4. Plots of $\langle k \rangle$, as a function of $p \in [0,1]$, for infinite-length, finite-width self-dual strips of the square lattice. For a given p, in order of descending value of $\langle k \rangle$, the curves refer to are for $1_{\mathrm{sd}} \le (L_y)_{\mathrm{sd}} \le 4_{\mathrm{sd}}$.

III. STRIPS OF THE TRIANGULAR LATTICE

A. Free transverse boundary conditions

1. 2_F

The free energy for the Potts model on this strip is [8]

$$f(\mathrm{tri}, 2_F, q, v) = \frac{1}{2} \ln \lambda_{t2F} \tag{3.1}$$

where

$$\lambda_{t2F} = \frac{1}{2}[T_{t2F} + (3v + v^2 + q)\sqrt{R_{t2F}}] \tag{3.2}$$

with

$$T_{t2F} = v^4 + 4v^3 + 7v^2 + 4qv + q^2 \tag{3.3}$$

and

$$R_{t2F} = q^2 + 2qv - 2qv^2 + 5v^2 + 2v^3 + v^4. \tag{3.4}$$

From this we calculate

$$\langle k \rangle_{\mathrm{tri},2_F} = \frac{(1-p)^3}{1-p+p^2}. \tag{3.5}$$

Note that this expression for $\langle k \rangle$ is the same as that for the $L_y=1$ self-dual strip in Eq. (2.34). This provides an illustration of the fact that two different families of lattice strips may have the same average cluster number $\langle k \rangle$. We plot this cluster number $\langle k \rangle$ in Fig. 5, together with the cluster numbers for the various strips of the triangular lattice with greater widths and free or periodic boundary conditions. The values of $\langle k \rangle$ for $p = p_{c,\mathrm{tri}}$ are listed in Table I. The Taylor series expansions of Eq. (3.5) for $p \to 0$ and $r = 1 - p \to 0$ are the same as those of Eq. (2.34).

2. $3_F, 4_F, 5_F$

The free energy f for the strips of the triangular lattice with width $L_y = 3, 4, 5$ and free transverse boundary conditions were computed in Ref. [15] (see also [16]). We have

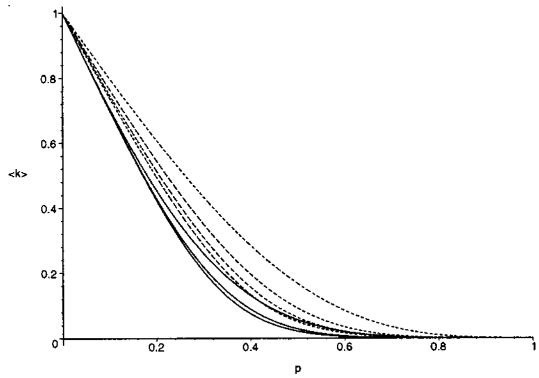

FIG. 5. Plots of $\langle k\rangle$, as a function of $p\in[0,1]$, for infinite-length, finite-width strips of the triangular lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k\rangle$, for $2_F \leqslant (L_y)_F \leqslant 5_F$, and the solid curves are, in the same order, for $2_P \leqslant (L_y)_P \leqslant 4_P$.

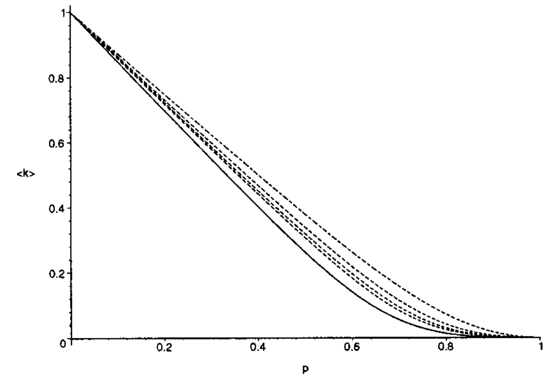

FIG. 6. Plots of $\langle k\rangle$, as a function of $p\in[0,1]$, for infinite-length, finite-width strips of the honeycomb lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k\rangle$, for $2_F \leqslant (L_y)_F \leqslant 5_F$, and the solid curve is for 4_P.

used these exact analytic expressions to obtain high- precision numerical computations of $\langle k\rangle$ for these strips.

B. Periodic transverse boundary conditions

2_P

Having explained our calculational method above for the square-lattice and previous triangular-lattice strips, we omit the details for other lattice strips except where we have carried out new calculations of Potts model free energies. The free energy $f(\mathrm{tri},2_P,q,v)$ was calculated in Ref. [8]. From it we compute

$$\langle k\rangle_{\mathrm{tri},2_P} = \frac{(1-p)^4(2+2p-7p^2+4p^3-p^4+2p^5-p^6)}{2(1-2p^2+8p^3-12p^4+8p^5-2p^6)}. \tag{3.6}$$

This has the respective Taylor series expansions for $p\to 0$ and $p\to 1$:

$$\langle k\rangle_{\mathrm{tri},2_P} = 1-3p+\frac{1}{2}p^2+4p^3+\frac{9}{2}p^4-10p^5-10p^6+O(p^7), \tag{3.7}$$

$$\langle k\rangle_{\mathrm{tri},2_P} = \frac{1}{2}r^4+2r^6-2r^8+\frac{9}{2}r^{10}+O(r^{11}). \tag{3.8}$$

The poles of $\langle k\rangle$ occur at

$$p \simeq -0.3744,\quad 1.6539,\quad 0.1731\pm 0.6306i,\quad 1.1872\pm 0.6924i. \tag{3.9}$$

The first two poles, lying on the real- p axis, are closest to the points $p=0$ and $p=1$ and determine the radii of convergence of the series about these points to be approximately 0.3744 and 0.6539, respectively.

Our procedure for strips with greater widths $L_y \geqslant 3$ is as before for the square lattice and $(L_y)_F$ triangular-lattice strips. The small- p series for the tri, 3_P lattice is

$$\langle k\rangle_{\mathrm{tri},3_P} = 1-3p+\frac{7}{3}p^3+6p^4+O(p^5). \tag{3.10}$$

IV. STRIPS OF THE HONEYCOMB LATTICE

The free energy $f(\mathrm{hc},2_F,q,v)$ was calculated in Ref. [9]. From it we obtain

$$\langle k\rangle_{\mathrm{hc},2_F} = \frac{(1-p)^2(4+3p+2p^2+p^3-4p^4)}{4(1-p^4+p^5)}. \tag{4.1}$$

This has the respective Taylor series expansions

$$\langle k\rangle_{\mathrm{hc},2_F} = 1-\frac{5}{4}p+\frac{1}{4}p^6+\frac{1}{4}p^{10}-\frac{1}{4}p^{11}+O(p^{14}), \tag{4.2}$$

$$\langle k\rangle_{\mathrm{hc},2_F} = \frac{3}{2}r^2+3r^3+O(r^4). \tag{4.3}$$

The expression (4.1) has poles at

$$p \simeq -0.8567,\quad -0.150\,05\pm 0.8975i,\quad 1.0784\pm 0.4969i. \tag{4.4}$$

The first of these is the nearest to the point $p=0$, so that the small- p Taylor series converges for $|p|\lesssim 0.8567$. The last pair of complex-conjugate poles is closest to $p=1$, so that the series for $r\to 0$ converges for $|1-p|\lesssim 0.5031$.

Strips of the honeycomb lattice with other widths and boundary conditions are analyzed using the same techniques as discussed above. The resulting cluster numbers $\langle k\rangle$ are plotted in Fig. 6 and the values for $p=p_{c,\mathrm{hc}}$ are listed in Table I.

V. STRIPS OF THE KAGOMÉ LATTICE

A. 2_F

For the purpose of obtaining $\langle k\rangle$, we have carried out a calculation of the free energy of the Potts model on the 2_F

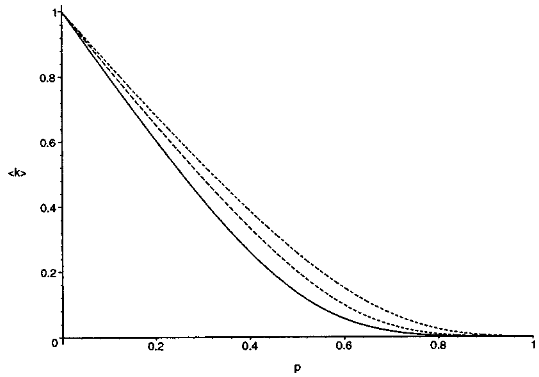

FIG. 7. Plots of $\langle k\rangle$, as a function of $p\in[0,1]$, for infinite-length, finite-width strips of the kagomé lattice. The dashed and solid curves refer to free and periodic transverse boundary conditions, respectively. For a given p, the dashed curves are, in order of descending value of $\langle k\rangle$, for $2_F \leqslant (L_y)_F \leqslant 3_F$, and the solid curve is for 2_P.

strip of the kagomé lattice. The results are sufficiently lengthy that we list them in the Appendix. From these we calculate

$$\langle k\rangle_{\text{kag},2_F} = \frac{N_{k2F}}{D_{k2F}} \tag{5.1}$$

where

$$N_{k2F} = (1-p)^2(5+2p-p^2-2p^3-8p^4-16p^5+43p^6-26p^7 - 2p^8+10p^9-3p^{10}-2p^{11}+p^{12}) \tag{5.2}$$

and

$$D_{k2F} = 5(1-p^4-2p^5+10p^6-10p^7+3p^8). \tag{5.3}$$

We plot $\langle k\rangle_{\text{kag},2_F}$ in Fig. 7. The value of $p_{c,\text{kag}}$ has been determined numerically [23] as $p_{c,\text{kag}}=0.524\,405\,3(3)$; at this value of p, our expression (5.1) for $\langle k\rangle_{\text{kag},2_F}$ has the approximate value 0.229 18. The $\langle k\rangle$ for this 2_F kagomé strip has the following expansions in the vicinity of $p=0$ and $p=1$:

$$\langle k\rangle_{\text{kag},2_F} = 1 - \frac{8}{5}p + \frac{2}{5}p^3 + \frac{1}{5}p^6 + O(p^7), \tag{5.4}$$

$$\langle k\rangle_{\text{kag},2_F} = \frac{1}{5}r^2 + \frac{8}{5}r^3 + \frac{11}{5}r^4 - \frac{4}{5}r^5 + O(r^6). \tag{5.5}$$

The cluster number (5.1) has poles at

$$p \simeq -0.5470 \pm 0.2862i, \quad -0.0363 \pm 0.6583i, \quad 0.7772 \pm 0.5605i, \quad 1.4728 \pm 0.1486i. \tag{5.6}$$

Of these, the first and last complex-conjugate pairs are closest to $p=0$ and $p=1$, respectively, and determine the radii of convergence of the Taylor series expansions about these points to be approximately 0.6174 and 0.4956.

We have also calculated $\langle k\rangle$ for the 3_F strip of the kagomé lattice; this is plotted in Fig. 7.

B. 2_P

For the 2_P strip of the kagomé lattice we find

$$\langle k\rangle_{kag,2_P} = \frac{N_{k2P}}{D_{k2P}} \tag{5.7}$$

where

$$N_{k2P} = (1-p)^4(6+12p+12p^2+4p^3-25p^4-108p^5+16p^6 + 472p^7-706p^8+320p^9+286p^{10}-352p^{11}-194p^{12} + 360p^{13}+120p^{14}-340p^{15}+65p^{16}+136p^{17}-96p^{18} + 24p^{19}-2p^{20}) \tag{5.8}$$

and

$$D_{k2P} = 6(1-2p^4-8p^5+32p^6+40p^7-268p^8+424p^9 - 320p^{10}+120p^{11}-18p^{12}). \tag{5.9}$$

A plot is given in Fig. 7. The cluster number $\langle k\rangle_{\text{kag},2_P}$ has poles at

$$p \simeq -0.4660, \quad 1.6556, \quad -0.3443 \pm 0.2919i, \quad -0.0057 \pm 0.4751i, \quad 0.5325 \pm 0.484\,55i, \quad 1.0776 \pm 0.4384i, \quad 1.4785 \pm 0.2140i. \tag{5.10}$$

The first complex-conjugate pair is the nearest to the origin and sets the radius of convergence of the small- p Taylor series expansion of $\langle k\rangle_{\text{kag},2_P}$ as 0.4514, while the second-to-last complex-conjugate pair is closest to the point $p=1$ and determines the radius of convergence of the series expansion about this point to be 0.4453, to the stated accuracy.

To our knowledge, it is not known what the value of $\langle k\rangle$ is for the (infinite) kagomé lattice at the numerically determined critical percolation probability $p_{c,\text{kag}}$. Assuming that, for a given set of transverse boundary conditions and a given $p\in(0,1)$, $\langle k\rangle$ is a monotonically decreasing function of the strip width L_y for this lattice, as we find for other lattice strips, our results yield the upper bound $\langle k\rangle_{\text{kag}} < \langle k\rangle_{\text{kag},2_P} \simeq 0.111\,49$ at the value $p=p_{c,\text{kag}}$ given above. Here we use the result for the 2_P strip since it is lower than the result for the 2_F and 3_F strips.

VI. DISCUSSION

We first introduce a notion of effective coordination number. For a graph G the degree of a vertex is the number of bonds connected to this vertex. A κ-regular graph is a graph in which all of the vertices have the same degree κ. Whether a given lattice strip graph is κ-regular depends on the longitudinal and transverse boundary conditions; for example, it is κ-regular if one uses toroidal (doubly periodic) boundary conditions. In the limit $L_x \to \infty$, since the longitudinal boundary conditions do not affect the free energy $f(\{G\},q,v)$, we need only consider the effect of the transverse boundary conditions. The effective coordination number is

$$\kappa_{\rm eff}(\{G\}) = \lim_{n\to\infty} \frac{2e(G)}{n(G)}. \qquad (6.1)$$

Clearly $\kappa_{\rm eff}=\kappa$ for a regular lattice. For regular lattice strips with periodic transverse boundary conditions, the value of $\kappa_{\rm eff}$ is the same as the value for the corresponding two-dimensional lattice. For strips with free transverse boundary conditions, we have

$$\kappa_{\rm eff}(\Lambda,(L_y)_F) = \kappa_\Lambda\left(1-\frac{\alpha}{L_y}\right) \qquad (6.2)$$

where $\kappa_\Lambda=4,6,3$ for $\Lambda=\text{sq},\text{tri},\text{hc}$ and

$$\alpha_{\rm sq}=\frac{1}{2}, \quad \alpha_{\rm tri}=\frac{2}{3}, \quad \alpha_{\rm hc}=\frac{1}{3}. \qquad (6.3)$$

For the cyclic self-dual strips of the square lattice, the single external vertex connected to each of the sites on one side of the strip has a degree L_x that diverges as $L_x\to\infty$. The $L_x(L_y-1)$ interior vertices have degree 4, while the L_x vertices on the rim have degree 3. Together, these lead, in the limit $L_x\to\infty$, to the result $\kappa_{\rm sq,sd}=4$. Finally, for the kagomé strips with free transverse boundary conditions

$$\kappa_{\rm eff}(\text{kag},(L_y)_F) = 4\left(1-\frac{1}{3L_y-1}\right), \qquad (6.4)$$

while for the kagomé strips with periodic transverse boundary conditions, $\kappa_{\rm eff}=4$, the same value as for the infinite two-dimensional kagomé lattice.

From our calculations we find a number of generic features.

(1) We have shown that $\langle k\rangle$ is a (real) analytic function of p in the interval $0\le p<1$. At the critical percolation probability $p=1$ for these quasi-1D strips, our exact results for $\langle k\rangle$ are also analytic, although some other quantities in percolation, such as the percolation probability $P(p)$ and the cluster size $S(p)$ are not, as is evident from the well-known 1D case.

(2) As the curves in the figures show, with an increase in strip width L_y, $\langle k\rangle$ is consistent with approaching a limiting function of p. This is in accord with one's expectation.

(3) For a given p in the interval between 0 and 1, and for a given type of lattice strip, as the width L_y increases, $\langle k\rangle$ decreases, so that the approach to the asymptotic value for the 2D lattice is from above, in the cases that we have computed. For strips with free transverse boundary conditions, increasing L_y increases $\kappa_{\rm eff}$, so the decrease of $\langle k\rangle$ is associated with an increase in the effective coordination number. This is reasonable, since, heuristically, for a fixed value of p, there is a greater probability of having a percolating cluster on a lattice of higher coordination number, so that more sites are part of this cluster and there are fewer separate clusters per site. This is also reflected in the monotonic decrease of $p_{c,\Lambda}$ with increasing κ_Λ for most higher-dimensional lattices. (However, we recall that counterexamples to this general monotonic decrease of $\langle k\rangle$ with increasing coordination number are known [26].)

For strips with periodic transverse boundary conditions, the decrease of $\langle k\rangle$ at a fixed p with increasing width L_y is not associated with an increase in $\kappa_{\rm eff}$, since $\kappa_{\rm eff}$ is constant for these strips (and equal to the two-dimensional value); here one may interpret the decrease as being simply due to a reduction in the finite-width effects that enables the percolation quantities to approach their two-dimensional values.

(4) For a given lattice type, we find some examples where the curve for $\langle k\rangle$ calculated on a strip of width L_y with periodic transverse boundary conditions will cross the curve for $\langle k\rangle$ for the same lattice and a different L_y and free transverse boundary conditions. For example, as is evident in Fig. 1, the curve for $\langle k\rangle$ on the $\text{sq},2_P$ strip lies below those for $\langle k\rangle$ on the $\text{sq},(L_y)_F$ strips at small p, but sequentially crosses the latter as p increases and lies above them (except for $L_y=1,2$) as $p\to 1^-$. Similar behavior is observed, e.g., on the strips of the triangular lattice. These also constitute examples of how $\langle k\rangle$ calculated on a strip with a larger value of $\kappa_{\rm eff}$ than that of another strip can be larger than $\langle k\rangle$ for the latter strip. For instance, $\kappa_{\rm eff}=\kappa=4$ for the $\text{sq},2_P$ strip, which is larger than the value $\kappa_{\rm eff}=3.6$ for the $\text{sq},5_F$ strip; however, $\langle k\rangle$ on the former strip is larger than $\langle k\rangle$ on the latter for $p\gtrsim 0.36$. This dependence on transverse boundary conditions is consistent with disappearing as the strip width $L_y\to\infty$, consistent with the approach to a single limiting function $\langle k\rangle$ for the corresponding 2D lattice. Although we have not proved rigorously that the function $\langle k\rangle$ obtained via this limiting sequence (taking $L_x\to\infty$ first and then taking $L_y\to\infty$) is identical to the function $\langle k\rangle$ obtained via the usual two-dimensional thermodynamic limit ($L_x\to\infty$, $L_y\to\infty$ with L_y/L_x a nonzero finite number), this conclusion is consistent with our findings.

(5) We have used the values of $\langle k\rangle$ at $p=p_{c,\Lambda}$ as a measure of how rapidly, for a given p, the cluster number calculated on infinite-length, finite-width strips approaches the value for the two-dimensional lattice. These values are listed in Table I. Even for the modest strip widths considered here, one sees that (i) these values approach the known values of $\langle k\rangle$ on the corresponding two-dimensional lattices reasonably quickly, and (ii) this approach is more rapid when one uses periodic transverse boundary conditions, as is expected, since the latter minimize finite-width effects. For example, for the strip of the square lattice with $L_y=5$ and periodic transverse boundary conditions, $\langle k\rangle$ evaluated at $p=p_{c,\rm sq}$ is about 15% larger than $\langle k\rangle_c$ for the square lattice [24], while $\langle k\rangle$ for the $\text{tri},4_P$ and $\text{hc},4_P$ strips, evaluated at the respective $p_{c,\rm tri}$ and $p_{c,\rm hc}$, are both about 17% larger than the corresponding values [24] $\langle k\rangle_c$ for the triangular and honeycomb lattices.

(6) We find that for these strips, the small-p series expansions of $\langle k\rangle$ have the leading terms

$$\langle k\rangle = 1-\left(\frac{\kappa_{\rm eff}}{2}\right)p+\cdots, \qquad (6.5)$$

which are analogous to the structure that these series have for regular lattices of dimension $d\ge 2$. Higher-order terms in the series for the strips of small widths are not expected to coincide with those in the series for the two-dimensional lattices, and one sees that they do not.

(7) An interesting output of our analysis is the exact determination, for various infinite-length, finite-width strips, of the singularities of $\langle k\rangle$ in the complex- p plane. As we have shown, for many strips these (real and/or complex) singularities outside the physical interval $[0,1]$ occur sufficiently close to the points $p=0$ and $p=1$ that they render the radii of convergence of the respective Taylor series expansions about these points less than unity, although the actual functions $\langle k\rangle$ themselves are analytic functions on $p\in[0,1]$. Although the strip widths are probably too small to justify a detailed comparison with unphysical singularities for percolation quantities in two dimensions, this generic property—the presence of unphysical singularities that determine the radii of the Taylor series expansions about the points $p=0$ and $p=1$ to be less than p_c for the given type of lattice—is similar to what was found in analyses of series for the percolation problem on two- and three-dimensional lattices [1].

(8) Finally, we have discussed how, for a given infinite-length, finite-width strip, the unphysical singularities have a connection with the locus $\mathcal{B}_{qn}$, which is the continuous accumulation set of the zeros of the Potts model partition function in the p (or equivalently the v) plane obtained by first letting $n\to\infty$ and then $q\to 1$. In particular, we find that these unphysical singularities lie on $\mathcal{B}_{qn}$. The noncommutativity of Eq. (2.11) analyzed in Ref. [6] plays a crucial role here, since $\mathcal{B}_{nq}$, obtained by first letting $q\to 1$ and then $n\to\infty$, is trivial. Our results motivate further study on this topic.

VII. CONCLUSIONS

In summary, we have presented exact calculations of the average cluster number per site $\langle k\rangle$ for the bond percolation problem on infinite-length, finite-width strips of the square, triangular, honeycomb, and kagomé lattices, with both free and periodic transverse boundary conditions. We believe that these results are a useful extension beyond the one-dimensional result toward two dimensions and provide insight into the form of $\langle k\rangle$ as a function of the bond occupation probability p.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we give the free energy for the Potts model on the 2_F strip of the kagomé lattice. We find

$$f(\mathrm{kag},2_F,q,v)=\frac{1}{5}\ln\lambda_{k2F} \tag{A1}$$

where

$$\lambda_{k2F}=\frac{1}{2}[T_{k2F}+\sqrt{R_{k2F}}] \tag{A2}$$

with

$$T_{k2F}=v^8+8v^7+v^6q+29v^6+20v^5q+10v^4q^2+2v^3q^3 + 42v^5+61v^4q+54v^3q^2+28v^2q^3+8vq^4+q^5 \tag{A3}$$

and

$$\begin{aligned}R_{k2F}={}&v^{16}+16v^{15}+2v^{14}q+114v^{14}+32v^{13}q-3v^{12}q^2-4v^{11}q^3+484v^{13}+288v^{12}q+52v^{11}q^2-28v^{10}q^3-12v^9q^4-2v^8q^5\\&+1329v^{12}+1572v^{11}q+1098v^{10}q^2+520v^9q^3+192v^8q^4+48v^7q^5+6v^6q^6+2196v^{11}+4350v^{10}q+5196v^9q^2+4344v^8q^3\\&+2628v^7q^4+1114v^6q^5+312v^5q^6+52v^4q^7+4v^3q^8+1620v^{10}+4572v^9q+7413v^8q^2+8284v^7q^3+6732v^6q^4\\&+4028v^5q^5+1766v^4q^6+556v^3q^7+120v^2q^8+16vq^9+q^{10}.\end{aligned} \tag{A4}$$

[1] J. W. Essam, Rep. Prog. Phys. **43**, 833 (1980).
[2] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor and Francis, London, 1991).
[3] G. Grimmett, *Percolation*, 2nd ed. (Springer, New York, 1999).
[4] This is a shortened version of S.-C. Chang and R. Shrock, e-print cond-mat/0407070.
[5] Here the partition function of the q-state Potts model on a graph G at temperature $1/K$ is $Z(G,q,v)=\Sigma_{G'\subseteq G}\, q^{k(G')}v^{e(G')}$, where $v=e^K-1$, G' is a spanning subgraph of G, and $k(G')$ and $e(G')$ denote the numbers of connected components and edges of G'.
[6] R. Shrock, Physica A **283**, 388 (2000).
[7] S.-C. Chang and R. Shrock, Physica A **296**, 234 (2001).
[8] S.-C. Chang and R. Shrock, Physica A **286**, 189 (2000).
[9] S.-C. Chang and R. Shrock, Physica A **296**, 183 (2001).
[10] S.-C. Chang and R. Shrock, Physica A **301**, 196 (2001).
[11] S.-C. Chang and R. Shrock, Int. J. Mod. Phys. B **15**, 443 (2001).
[12] S.-C. Chang and R. Shrock, Physica A **301**, 301 (2001).
[13] S.-C. Chang and R. Shrock, Phys. Rev. E **64**, 066116 (2001).
[14] S.-C. Chang, J. Salas, and R. Shrock, J. Stat. Phys. **107**, 1207 (2002).
[15] S.-C. Chang, J. Jacobsen, J. Salas, and Shrock, J. Stat. Phys.

114, 763 (2004).
[16] S.-C. Chang and R. Shrock, e-print cond-mat/0404524.
[17] The values of $p_{c,\Lambda}$ for the square (sq), triangular (tri), and honeycomb (hc) lattices are known exactly: $p_{c,\mathrm{sq}}=1/2$, $p_{c,\mathrm{tri}}=2\sin(\pi/18)\simeq 0.347\,296$, and $p_{c,\mathrm{hc}}=1-p_{c,\mathrm{tri}}\simeq 0.652\,703\,6$ [1,2].
[18] For periodic longitudinal boundary conditions, the partition function for the $\mathrm{sq},2_F$ strip involves six λ's [6], but since the free energy is independent of the longitudinal boundary conditions in the case $L_x\to\infty$ of interest, it will suffice to consider only the case of free longitudinal boundary conditions.
[19] R. Shrock and S.-H. Tsai, Phys. Rev. E **55**, 5165 (1997).
[20] G. Marchesini and R. Shrock, Nucl. Phys. B **318**, 541 (1989).
[21] V. Matveev and R. Shrock, J. Phys. A **28**, 1557 (1995); **28**, 4859 (1995); **28**, 4859 (1995); **28**, L533 (1995); Phys. Lett. A **204**, 353 (1995); J. Phys. A **29**, 803 (1996); Phys. Rev. E **54**, 6174 (1996).
[22] H. Feldmann, R. Shrock, and S.-H. Tsai, Phys. Rev. E **57**, 1335 (1998); H. Feldmann, A. Guttmann, I. Jensen, R. Shrock, and S.-H. Tsai, J. Phys. A **31**, 2287 (1998).
[23] R. Ziff and P. Suding, J. Phys. A **30**, 5351 (1997).
[24] Known values of $\langle k\rangle_c$ are [25] $\langle k\rangle_{c,\mathrm{sq}}=(3^{3/2}-5)/2\simeq 0.098\,076\,2$ and $\langle k\rangle_{c,\mathrm{tri}}=35/4-3/p_{c,\mathrm{tri}}\simeq 0.111\,844$; taking into account that $\langle k\rangle$ is defined per site, while n_c^{B-HC} in [25] is defined per unit cell and there are two sites per unit cell on the honeycomb lattice, it follows that $\langle k\rangle_{c,\mathrm{hc}}=(1/2)(\langle k\rangle_{c,\mathrm{tri}}+p_{c,\mathrm{tri}}^3)\simeq 0.076\,866\,67$.
[25] R. Ziff, S. Finch, and V. Adamchik, Phys. Rev. Lett. **79**, 3447 (1997).
[26] S. van der Marck, Phys. Rev. E **55**, 1514 (1997); **56**, 3732(E) (1997); J. Wierman, *ibid.* **66**, 046125 (2002).